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Powerlike Decreasing Solutions of the Boltzmann Equation for a Maxwell Gas

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We study the homogeneous, isotropic, nonlinear Boltzmann equation for a Maxwellian interaction. We show that solutions decreasing like inverse powers of the energy are physically acceptable both in the linearized and the quadratic problem. Because all moments may not exist, we introduce a generalized generating function and a finite differential system for generalized Sonine moments is derived. These new solutions may lead to small relaxation rates and justify in most cases the linear approximation.

KEY WORDS: Nonequilibrium statistical mechanics; nonlinear Boltzmann equation; Maxwellian interaction; energy power like decreasing solutions; relaxation rates; Tjon's effect.

1. INTRODUCTION

Within the last years, much work has been done on the homogeneous, isotropic solutions of the nonlinear Boltzmann equation (B.E.) when the interaction between particules are Maxwellian, i.e., $V(r) \sim r^{-4}$. The main reason is that the eigenfunctions of the *linearized* problem are easy to handle and yield for the full *nonlinear* problem a convenient basis where the problem reduces to a recursive finite differential system in the time variable.

The process, first developed by Wu *et al.*,[1-4] after Grad's method,[5] is the following: one first derives from the B.E. a finite nonlinear differential system for the moments of the distributions function; this system equally holds for the Sonine moments which are linearly related to them. It remains to prove, that if the expansion has a meaning at initial time, it is still true at further times and that it converges to the equilibrium function e^{-v^2} for long times.[6-8]

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The eigenfunctions $L_n^{(d/2-1)}(x)e^{-x}$ [d is the dimensionality, $x = v^2$ is the energy variable, and $L_n^{(\alpha)}(x)$ is the Laguerre polynomial] have two important characteristics: (i) they decrease exponentially for large energies, and (ii) they are a basis of the Hilbert space $\mathcal{H}_I$ of the functions with norm $\int f^2(v,t)e^{+v^2}d\mathbf{v} < \infty$, and the Hilbert operator associated to the B.E. is square integrable in this space. Thus all these eigenfunctions span a rather large space, and reasonably one may hope that most initial distribution functions belong to it.

One may wonder whether we find this to be so for all solutions of the Boltzmann equation. The answer is no in the linearized case, where Bobylev[9] found other solutions which were not decreasing exponentially, but like inverse powers of the energy variable; they no more belong to the Hilbert space $\mathcal{H}_I$ as their norm is infinite. One problem is to see whether they are relevant for the full nonlinear equation[10] and, in the affirmative case, how all previous results are modified.

In the present paper and in a preceding letter[11] we try to answer these questions for a Maxwellian gas with isotropic cross section for the $3d$ case and the corresponding $2d$ Tjon–Wu model[3]; generalization to all dimensions d is equally indicated. Bobylev's new eigenfunctions lead again to solutions of the quadratic problem but they generate a larger Hilbert space $\mathcal{H}_{II}$ with another norm. They cannot be rejected as they satisfy the physical requirements of mass and energy conservation and of finite entropy, but as the Hilbert operator is no more square integrable, they are not independent.[11] Notice that, in the linearized case for a hard-sphere gas,[12] the solutions are not acceptable and presumably Maxwell interaction corresponds to a limit situation.

As to the derivation of the differential system, Grad's method is no more valid because moments may not exist. We define a generalized generating function which reduces to the ordinary generating functional of the power moments when the distribution function belongs to $\mathcal{H}_I$. This generalized generating function still has a meaning when either some of the moments or the ordinary generating function do not exist. It satisfies a nonlinear partial differential equation (N.L.P.D.E.)—and not an integro-differential equation—which has remarkable symmetry and invariance properties and is dimensionally independent; in the Laguerre case, the N.L.P.D.E. was first established by Krook and Wu.[1,2] The eigenfunctions follow easily and we derive a recursive differential system for generalized Sonine moments: a possible generalization of the ordinary moments is given. Notice that we do not retain all found solutions of the N.L.P.D.E. as the other possibilities must be rejected because of the physical restrictions.

In Section 3.1, proof for the convergence of generalized expansions is provided, and we obtain sufficient conditions at $t = 0$ such that the norm of the solution in the generalized Hilbert space exists when $t \neq 0$ and $t \to \infty$, where it reduces to the Maxwellian form. We show how to generate infinite sets of positive initial distributions; they remain positive at further times.

In Section 3.2, we show that there are infinitely many similarity solutions, the Bobylev(13),2 Krook–Wu solution(1,2) being not the general situation; nevertheless, it seems difficult to prove that some of them are positive, and the one-parameter Krook–Wu family is up to now the only one which possesses good properties for a distribution function.

In Section 4, we give numerical results. Because of the asymptotic behavior of the eigenfunctions at large energy, numerical effects which were important in the Laguerre case are modified: for example, Tjon's overpopulation(14) is true only for intermediate times or intermediate energies. On the other hand, new effects take place. The most interesting is the possibility of very slow relaxation to equilibrium as relaxation rates are not bounded from below; actually slow relaxation follows from the slow powerlike decrease of the eigenfunctions at large energies and conversely. Another feature is the validity of the linearized form for long times in rather general situations. Several examples are given.

2. N.L.P.D.E. FOR THE GENERALIZED GENERATING FUNCTION

As explained in the Introduction, we shall not keep the energy variable $v^2 = x$, as the Boltzmann equation depends on dimensionality and (except for $d = 2$) in a complicated way of $\mathbf{v}$ through the collision velocities $\mathbf{v}'$ and $\mathbf{w}'$. We look for a transformed function $G(p, t)$ of a unique conjugate variable p, which should be dimensionally universal and should satisfy a more tractable N.L.P.D.E. so that all results would be simpler.

2.1. Definition of $G(p,t)$

A "natural" definition is

$$G(p,t) = \pi^{-d/2} \int \phi\left(1, \frac{d}{2}\ ; -pv^2\right) f(v,t)\, d\mathbf{v} \tag{1a}$$

$$= \left[\Gamma\left(\frac{d}{2}\right)\right]^{-1} \int_0^\infty x^{d/2-1} \phi\left(1, \frac{d}{2}\ ; -px\right) F(x,t)\, dx \tag{1b}$$

where the isotropic and homogeneous distribution function $f(v,t) = F(x,t)$ is rewritten in terms of the energy variable, d is the dimensionality, and ϕ the confluent hypergeometric function; constants $\pi^{-d/2}$ or $\Gamma^{-1}(d/2)$ are introduced for further simplification of Eq. (3). When all the reduced

[2] Similarity solutions for related models have been studied in Ref. 13b.

moments $M_n = \int d\mathbf{v}\, v^{2n} f(v,t) / \int d\mathbf{v}\, e^{-v^2} v^{2n}$ of $F(x,t)$ exist, $G(p,t)$ is simply their ordinary generating function $G(p,t) = \sum_n (-p)^n M_n$ (if the sum has a meaning) as introduced by Krook and Wu.[1,2] But when moments become infinite, or their ordinary generating function does not converge, as will be the case here, $G(p,t)$ is defined directly by Eq. (1a) or (1b), provided that the integral converges, even for $p = 0$; this yields $f(v,t) = o_{v\to\infty}(v^{-(d+\epsilon)})$ $(\epsilon > 0)$. If besides we impose the existence of the second moment M_1 (the energy) we get the asymptotic condition

$$f(v,t) = \underset{v\to\infty}{o}(v^{-(2+d+\epsilon)}), \qquad \epsilon > 0 \tag{2}$$

which is far less restrictive than the usual exponential decrease requirement.

2.2. The Equation for G

When all moments exist and $d = 2, 3$, it was proved[1,2] that $G(p,t)$ is a solution of the simple N.L.P.D. equation

$$\frac{\partial}{\partial p}\left(\frac{\partial}{\partial t} + 1\right) pG(p,t) = G^2(p,t) \tag{3}$$

and this result may be extended easily to the d-dimensional case. When moments become infinite, we deduce Eq. (3) directly from the Boltzmann equation. In the Tjon–Wu $2d$ model, $G(p,t)$ is the Laplace transform in the energy variable. From

$$\left(\frac{\partial}{\partial t} + 1\right) F(x,t) = \int_x^\infty \frac{dx'}{x'} \int_0^{x'} dx''\, F(x' - x'', t) F(x'', t) \tag{4}$$

we get

$$\left(\frac{\partial}{\partial t} + 1\right) G = \int_0^\infty dx''\, F(x'',t) \int_{x''}^\infty dx'\, F(x' - x'', t)\left(\frac{1 - e^{-px'}}{px'}\right)$$
$$= \frac{1}{p} \int_0^p G^2(p',t)\, dp'$$

whence (3) by simple derivation. As $F \simeq_{x\to\infty} x^{-1-\epsilon}$, Eq. (4) exists and all intermediate steps are allowed also.

For $d > 2$, it is more complicated and the proof is given in Appendix A. We sketch here the main steps. For $d = 3$ the B.E. reads

$$\left(\frac{\partial}{\partial t} + 1\right) f = N \int d\mathbf{w} \int \frac{d\Omega}{4\pi} f(v',t) f(w',t) \tag{5}$$

where N is the normalization factor $N^{-1} = \int e^{-w^2} d\mathbf{w} = \pi^{-3/2}$, when the equilibrium function is e^{-v^2}, $\mathbf{v}'$ and $\mathbf{w}'$ being the velocities after collision.

From (1a)

$$\left(\frac{\partial}{\partial t}+1\right)G=\frac{2}{\pi^2}\int_0^\infty dv\, v^2 f(v,t)\int_0^\infty dw\, w^2 f(w,t) I(p,v,w)$$

where

$$I(p,v,w)=\sum_{n\geqslant 0}\lambda_n(-p)^n\int_0^\pi \sin x\, dx\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\epsilon\, v'^{2n}$$

is an integral over diffusion angles χ, ϵ, and direction angle θ, and $\lambda_n=\Gamma(3/2)/\Gamma(n+3/2)$. Using Krook–Wu's techniques,[2] we rewrite I as a simple integral

$$I=8\pi\int_0^1 du\,\phi\left(1,\tfrac{3}{2},-upv^2\right)\phi\left(1,\tfrac{3}{2},-upw^2\right)$$

whence

$$\begin{aligned}\left(\frac{\partial}{\partial t}+1\right)G&=\frac{16}{\pi}\int_0^1 du\int_0^\infty v^2 f(v,t)\phi\left(1,\tfrac{3}{2},-upv^2\right)dv\\&\quad\times\int_0^\infty w^2 f(w,t)\phi\left(1,\tfrac{3}{2},-upw^2\right)dw\\&=\int_0^1 G^2(up,t)\,du\end{aligned}$$

and the announced result.

2.3. Properties of the Solutions of Eq. (3)

From now on, we shall work with Eq. (3), as it is simpler and d universal. In the conjugate space the condition for the conservation of the two first moments (mass and energy) reads

$$G(p,t)|_{p=0}\equiv 1 \tag{6a}$$

$$\left.\frac{\partial G(p,t)}{\partial p}\right|_{p=0}\equiv -1 \tag{6b}$$

their existence being insured by (2), and the relaxation to the Maxwell form e^{-v^2} implies

$$G(p,t)\underset{t\to\infty}{\to}\frac{1}{p+1} \tag{7}$$

2.3.1 The Linearized Case.

When looking for small deviations from the equilibrium, we set $G(p,t)=1/(p+1)+\overline{G}(p,t)$ and replace (3) by its linearized part

$$\frac{\partial}{\partial p}\left(\frac{\partial}{\partial t}+1\right)p\overline{G}=\frac{2}{p+1}\overline{G} \tag{8}$$

The eigenfunctions of the linear partial derivative equation are chosen of the form $\overline{G}=d(t)g(p)$, where the dependence on the two variables is factorized. Equation (8) is replaced by the two differential equations

$$(p+1)\left(\frac{p}{g}\frac{dg}{dp}+1\right)=\frac{2d(t)}{\dot{d}(t)+d(t)}=a+1$$

or

$$\begin{aligned}\frac{1}{g}\frac{dg}{dp}&=\frac{a+1}{p(p+1)}-\frac{1}{p}\\ (a+1)\dot{d}(t)+(a-1)d(t)&=0\end{aligned} \tag{9}$$

where $\dot{d}(t)$ is the time derivative of $d(t)$ and a is an arbitrary constant. It follows that

$$d_a(t)=\exp\left[-\left(\frac{a-1}{a+1}\right)t\right]d_a(0)$$

decreases exponentially with time when $a>1$ and the eigenfunctions are

$$g_a(p)=p^a/(p+1)^{a+1} \tag{10}$$

whence the general solution of (8),

$$G(p,t)=\frac{1}{p+1}+\sum_a d_a(t)\frac{p^a}{(p+1)^{a+1}} \tag{11a}$$

where the summation runs over any set $\{a\}$ even continuous. Mass and energy conservation conditions yield $a>0$ and $a>1$, respectively, and these inequalities precisely insure the existence of G [Eq. (2)] and an exponential decay for long time [Eq. (9)]. The summation in (11a) is thus restricted to

$$a>1 \tag{11b}$$

Provided $\operatorname{Re} p>-1/2$, we have $|p/p+1|<1$ and the convergence of (11a) is insured at least for countable sets $\{a\}$, every time $|d_a(t)|$ is uniformly bounded (in a and t). In Section 3.1.2, we prove finer results about the dependence on a and t for more restrictive conditions [Eq. (26b)]. Au.: see query msp. 260

In the energy space x, the eigenfunctions $g_a(p)$ correspond to the eigenfunctions

$$\psi_a(x)=\frac{\Gamma(a+d/2)}{\Gamma(a+1)\Gamma(d/2)}\phi\left(-a,\frac{d}{2},x\right)e^{-x} \tag{12}$$

where ϕ is the confluent (Kummer) hypergeometric function. The resulting velocity distribution function reads

$$F(x,t) = e^{-x} + \sum_{a>1} d_a(t)\psi_a(x) \tag{13}$$

and the eigenvalues are $\mu_a = (a-1)(a+1)$ for $a > 1$. The spectrum is no longer discrete but consists in the whole interval $]0,1[$. When a is an integer, say, n $(n \geqslant 2)$, the associate eigenfunction is $\psi_n = L_n^{(d/2-1)}(x)e^{-x}$ and we recover the well-known Laguerre solutions; moreover, if expansion (11a) contains only integer indices, the $d_n(t)$ are the ordinary Sonine moments.

But there are infinitely many other solutions, corresponding to a not an integer; the ψ_a's do not decrease exponentially for high energies but like inverse powers of x,

$$\psi_a \simeq x^{-(a+d/2)} \tag{14}$$

so that only the first moments M_0 and M_1 exist; the usual norm

$$\int f^2(v,t)e^{+v^2}\,d\mathbf{v}$$

is infinite and the associated Hilbert kernel is no more square integrable. This is the reason why they were long rejected though all physical conservation requirements were fulfilled. Another controversial point came from the wrong belief that finite norm yields finite entropy and conversely.[(15),3] Nevertheless, Bobylev[(9)] and recently Hauge and Praestgaard[(10)] noticed that they must be kept, at least in the linear case. As was said in a previous paper,[(11)] these solutions have a finite entropy and we have shown that they enter another mathematical frame: they generate a larger Hilbert space, with another norm; they are no more orthogonal and are overcomplete, i.e., they are not an independent set. The miracle is that all these solutions are still valid in the full nonlinear case as we shall see now.

2.3.2. The Nonlinear Case. As in the Laguerre case, we assume that the general solution of the linear case will be a guide to the full nonlinear problem. So, we look for solutions of the nonlinear equation (3), written as an expansion (11a, b). Putting (11a) in Eq. (3), we get

$$\sum_a \frac{p^a}{(p+1)^{a+1}}\left[(a+1)\dot{d}_a(t) + (a-1)d_a(t)\right]$$

$$= \sum_{b,b'>1} d_b(t)d_{b'}(t)\frac{p^{b+b'}}{(p+1)^{b+b'+1}}$$

[3] For a discussion see Ref. 15b.

If the set $\{a\}$ is closed, we have an obvious infinite differential system

$$(a+1)\dot{d}_a(t)+(a-1)d_a(t)=\sum_{\substack{b+b'=a\\ b,b'>1}} d_b(t)d_{b'}(t) \tag{15}$$

which is only possible when the $g_a(p)$ are linearly independent. System (15) may be derived more naturally by using (13) and the Laplace transform of the ψ_a for every dimension d, but the calculations happen to be more complicated. For $d=2$, it may be done directly by using the convolution properties of the hypergeometric (see the remark in Appendix A).

The great difference with the linear case arises from the necessity to make precise the set of indices $\{a\}$. To solve (15) in a recurrent way, it must be countable and closed. For $\{a\}=(2,3,4,\ldots,n,\ldots)$ we recover the Laguerre solutions, but if other sets $\{a\}$ may be found, it means that other solutions exist, which do not decrease exponentially at infinity, which have only a finite number of moments and an infinite norm but are physically perfectly acceptable. Moreover, as every eigenvalue $\mu_a=(a-1)/(a+1)$ may be reached by a suitable choice of the family $\{a\}$, it implies, as in the linear case, that there is no minimal relaxation rate.

We seek for a set of indices $1<a(0)<a(1)\ldots<a(n)\ldots$ such that

$$a(m)+a(m')=a(n)$$

whenever $m+m'\leqslant n$, for example, whenever $m+m'=n-P_0$, P_0 being a fixed integer.

The unique solution has the general form

$$a(n)=(P_0+n)/\xi, \qquad P_0 \text{ integer}, P_0>\xi>0 \tag{16a}$$

where ξ may be not rational. It may sometimes be convenient to include in the family the Maxwellian term $1/p+1$; then an alternate expression for $a(n)$ is

$$\tilde{a}(q)=q/\xi, \qquad \tilde{a}(q)=a(q-P_0) \tag{16b}$$

with $\tilde{a}(0)=0$. In the first case, the system (15) reads

$$(P_0+n+\xi)\dot{d}_n(t)+(P_0+n-\xi)d_n(t)=\xi\sum_{\substack{m+m'=n-P_0\\ m,m'>0}} d_m(t)d_{m'}(t) \tag{17a}$$

and in the second one

$$(q+\xi)\dot{\delta}_q(t)+(q-\xi)\delta_q(t)=\xi\sum_{\substack{r+r'=q\\ r,r'>0}}\delta_r(t)\delta_{r'}(t) \tag{17b}$$

provided that $\delta_0(t)=1$, $\delta_k(t)\equiv 0$ if $0<k<P_0$ and $\delta_q(t)=d_{q-P_0}(t)$. We shall use both forms.

The pure Laguerre case corresponds to $\xi=1$, $P_0=2$. More generally, if $\xi=N_0$ or $\xi=N_0/M_0$, N_0, M_0 integers, the family includes the $\psi_n(x)$

associated to Laguerre polynomials for $n \geqslant n_0 = 1 + [M_0 P_0 / N_0]$ ([ ] = integer part) and when $M_0 P_0 = N_0 + 1$, all the Laguerre come in. Then we have a good idea of the space generated, as it is an extension of the ordinary Hilbert space. But when ξ is not rational, the space spanned by the $\psi_{a(n)}$ is more difficult to handle. In Section 2.3.3 we shall see that this difference persists for some other points.

Remarks. (1) Other solutions of the N.L.P.D.E. may be derived by using the symmetry between variables t and $q = \ln p$. If $G(p,t)$ is a solution of (3), then $G(e^t, \ln p)$ is also a solution of (3); but it cannot be written as an expansion (11a) and does not converge to the Maxwell limit.

(2) Time t and energy v^2 play a nonsymmetric role in the Boltzmann equation. Symmetry can be put in evidence for an h function linked to G by using variables t and $u = \ln p/(p+1)$. In this way, we can obtain another expansion of G in terms of t-dependent functions with p-dependent coefficients. These coefficients satisfy a nonlinear differential system similar to (15) which can also be recursively solved (see Appendix B).

2.3.3. The Invariance Property. A natural variable for $G(p,t)$ is $u = p/(p+1)$ [or $p = u/(u+1)$] and expansions (11a) for $G(p,t)$ yield $H(u,t) = 1 + \sum_a d_a(t) u^a$, where we define the new function $H(u,t)$ through

$$pG(p,t) = uH(u,t) \tag{18}$$

Now, the N.L.P.D. equation (3) is obviously invariant in the change of variable $p \leftrightarrow u$ and of function $G \leftrightarrow H$,

$$\frac{\partial}{\partial u}\left(\frac{\partial}{\partial t} + 1\right) uH(u,t) = H^2(u,t)$$

and it is still true when u is replaced by $-u$. Then, *if there exists* a set $\{b\}$ of indices such that

$$uH(u,t) = \frac{u}{1-u}\left[1 + \sum_b \tilde{M}_b(t)\left(\frac{u}{1-u}\right)^b\right]$$

we have the following properties.

(i) The $\tilde{M}_b(t)$'s are the solution of an infinite differential system like (15):

$$(b+1)\frac{d}{dt}\tilde{M}_b(t) + (b-1)\tilde{M}_b(t) = \sum_{b'+b''=b} \tilde{M}_{b'}(t)\tilde{M}_{b''}(t)$$

(ii) We have the equation

$$G(p,t) = 1 + \sum_b \tilde{M}_b(t) p^b \tag{19}$$

(iii) If sets $\{a\}$ and $\{b\}$ are such that the expansion with set $\{a\}$ in the variable $p/(p+1)$ may be transformed in an expansion with set $\{b\}$ in variable p, then there are infinite linear relations between the sets $d_a(t)$ and $\tilde{M}_b(t)$. It is presumably not the case when one or both families are not generated by a rational. *When* ξ *is an integer* N_0, the two families are the same. Equations (11a) and (19) yield then [with $a(n) = n/N_0$, Eq. (16b)] the finite relations

$$\tilde{M}_n(t) = \sum_{m+kN_0=n} \frac{(-1)^k}{k!} d_m(t) \frac{\Gamma(n/N_0+1)}{\Gamma(m/N_0+1)} \tag{20a}$$

and

$$d_n(t) = \sum_{m+kN_0=n} \frac{\tilde{M}_m(t)}{k!} \frac{\Gamma(n/N_0+1)}{\Gamma(m/N_0+1)} \tag{20b}$$

When $N_0 = 1$, Eqs. (20) relate the Sonine moments d_n to the ordinary reduced moments $M_n = (-1)^n \tilde{M}_n$. It is interesting to look for a possible generalization for $N_0 > 1$. It follows from the definition (1b) and expansion (19) for G:

$$\Gamma\left(\frac{d}{2}\right) n! \tilde{M}_n = \lim_{\tilde{P}\to 0} \int_0^\infty dx\, x^{d/2-1} F(x,t) \frac{d^n}{d\tilde{P}^n} \phi\left(1, \frac{d}{2}, -\tilde{P}^{N_0} x\right)$$

where we have set $\tilde{P} = p^{1/N_0}$, but its interpretation is not easy to handle except in the $2d$ case where

$$n! \tilde{M}_n = \lim_{\tilde{P}\to 0} \int_0^\infty dx\, F(x,t) \frac{d^n}{d\tilde{P}^n} \exp(-\tilde{P}^{N_0} x)$$

if n is a multiple of N_0, say, $n = kN_0$, the eigenfunction ψ_{kN_0} is the Laguerre solution $\psi_k = e^{-x} L^{(d/2-1)}(x)$ and $(-1)^n \tilde{M}_n$ is the ordinary reduced moment M_k; when $n \neq kN_0$, the $\tilde{M}_n$ seems rather to be related to the asymptotic behavior of ψ_{n/N_0}. When $d = 2$ and $N_0 = 2$, $\tilde{M}_n$ can be expressed as

$$\tilde{M}_n = \frac{(-1)^n}{n!} \lim_{\tilde{P}\to 0} \int_0^\infty dx\, x^{n/2} H_n(\tilde{P}\sqrt{x}) \exp(-\tilde{P}^2 x) F(x,t)$$

where H_n is the nth Hermite polynomial.

3. PROPERTIES OF THE SOLUTIONS OF THE NONLINEAR DIFFERENTIAL SYSTEM (17)

For each family $\{a\}$, the recurrent differential system (17a, b) may be solved by the techniques used in previous publications.[6-8] The $d_n(t)$ are sums of functions decreasing exponentially in time and the corresponding solution $G(p,t)$ [respectively, $F(x,t)$] relaxes to the Maxwell limit $1/(p+1)$ [respectively, e^{-x}]. The proofs will be sketched in Section 3.1. We find that for n higher than some fixed critical value, the $d_n(t)$'s decrease at least like their linear part. This critical value becomes larger and larger when considering families of solutions with smaller and smaller relaxation rates. We define a Hilbert space $\mathcal{H}_{\text{II}}$ for the distribution functions corresponding to the deviation from equilibrium. We prove that the norms of these functions vanish when $t \to \infty$ at least like the first nonzero $d_n(t)$. We express also, in this space, in some well-defined way, the property that the contribution to the norm of the linear part of the d_n's is the dominant one when $t \to \infty$. We sketch briefly some properties of the solutions defined by the set $\{d_n(0)\}$; they are very useful for the control of the positivity property of $F(x, t)$.

A peculiar class of solutions, called similarity solutions (or fundamental solutions), occurs when the generalized Sonine moments $d_n(t)$ depend on only one relaxation rate. We study them in Section 3.2. We recover Bobylev[13] Krook–Wu solutions[2] but *there are infinitely many others* which do not have an exponential decrease at large energies. Unfortunately, it is very difficult to control their positivity.

3.1. Relaxation to Equilibrium

We must verify that the set of initial positive distribution functions $F(x, 0)$ which relax to the equilibrium Maxwell function e^{-x} is not empty—and actually infinite. Some slight differences with the pure Laguerre case appear but the central idea is the same. The proof and some complementary results are described in this section. The proof is divided into three parts:

3.1.1. Decrease in time for the d_n's. We start with general expression (16a) for $a(n)$, $a(n) = (n + P_0)/\xi$ and integrate system (17a):

$$d_n(t) = e^{-\gamma_n t}\left[d_n(0) + \frac{1}{\beta_n}\int_0^t e^{\gamma_n t'} \sum_{m+m'=n-P_0} d_m(t')d_{m'}(t')\right] \tag{21}$$

where

$$\gamma_n = (n + P_0 - \xi)/(n + P_0 + \xi) \tag{22a}$$

$$\beta_n = (n + P_0 + \xi)/\xi \tag{22b}$$

and $d_n(t)$ reduces to its linear part $d_n(0)e^{-\gamma_n t}$ whenever $0 \leqslant n \leqslant P_0 - 1$. It may be shown recursively that $d_n(t)$ decreases at least like $e^{-\gamma_n t}$. Notice that, as $\xi/P_0 \to 1$, γ_0 may become very small and the relaxation to equilibrium becomes very slow.

In the pure Laguerre case, a finer result[6] says that $d_n(t)$ decreases at least like $e^{-\gamma_n t}$. But, it is not always true in the general case. To be convinced of that, let $\xi = N_0 = 2$ and $P_0 = 3$; for $n = 3$, $\gamma_3 = 1/2$ but $d_3(t)$ behaves like $e^{-2/5t}$, and similarly $d_4(t)$ and $d_6(t)$ decrease, respectively, like

$e^{-8/15t}$ and $e^{-3/5t}$ and not like $e^{-5/9t}$ and $e^{-7/11t}$; for $n > 6$, $d_n(t) \sim e^{-\gamma_n t}$ holds. More generally, the number of coefficients $d_n(t)$ for which the behavior is not given by γ_n is finite, but it may be large and depends on the ratio $\rho = P_0/\xi$.

We have the following properties (proofs in Appendix C):

Proposition A. If

$$\rho > (1/4)(3+\sqrt{17}) \sim 1.7807$$

then $d_n(t)$ decreases exponentially like $e^{-\gamma_n t}$ for all n. Note that the pure Laguerre case corresponds to $\rho = 2$.

Proposition B. Let

$$\rho_k = (1/2k)\left[k + 1 + (k^2 + 6k + 1)^{1/2}\right], \qquad k = 2, 3 \ldots;$$

$$\rho_2 > \rho_3 \cdots > \rho_k \cdots > 1$$

then

$$\forall n \in \{n\}_k = \{(k-1)P_0 \leqslant n \leqslant kP_0 - 1\}$$

if $\rho > \rho_k$ then

$$d_n(t) \sim \exp\left[-\left(\frac{k\rho - 1}{k\rho + 1}\right)\right]t$$

if $\rho < \rho_k$ then

$$d_n(t) \sim \exp\left[-k\left(\frac{\rho - 1}{\rho + 1}\right)\right]t$$

Propositions A and B are only sufficient properties as each index n is considered a part of the block $\{n\}_k$ and is compared to the first index of the block, $(k-1)P_0$. The "good" behavior $e^{-\gamma_n t}$ is surely recovered, for the indices $n \geqslant (k_0 - 1)P_0$, k_0 being the first integer such that $\rho > \rho k_0$.

3.1.2. Sufficient Conditions for the Relaxation to Maxwell. In order to show that the set of solutions is not empty, we search for *sufficient* conditions on the initial distribution function so that $F(x,t)$ converges to Maxwell and is positive.[(6,8,16),4] Positivity will be treated in the next paragraph. Here we show that the norm of the deviation function $R(x,t) = F(x,t) - e^{-x}$ goes to zero for long times.

The usual inner product $\langle R_1, R_2 \rangle = \int_0^\infty x^{d/2-1} e^x R_1(x) R_2(x)\, dx$ being

[4] The positivity problem has also been discussed in Ref. 16.

infinite, we defined another one,[11]

$$\langle R_1, R_2\rangle = \int_0^{\infty} x^{d/2-1} R_1(x) R_2(x)\, dx \tag{23a}$$

and the norm is

$$\|R\|^2 = \langle R, R\rangle \tag{23b}$$

A basis is generated by the Laguerre functions $e^{-x} L_n^{(d/2-1)}(2x)$ of the argument $2x$ (and not x) and each $\psi_a(x)$ may be written[17]

$$\psi_a(x) = e^{-x} \frac{\Gamma(a+d/2)}{\Gamma(a+1)\Gamma(d/2)} \sum_{p=0}^{\infty} C_{p,a} L_p^{(d/2-1)}(2x)$$

where

$$C_{p,a} = \frac{(-1)^p 2^{-a} \Gamma(p-a)}{p!\Gamma(-a)}$$

is dimension independent. Using expansion (13) for $F(x,t)$, we get

$$\begin{aligned}\|R\|^2 = \sum_{b,b'} d_b(t) d_{b'}(t) &\frac{\Gamma(b+d/2)\Gamma(b'+d/2)}{\Gamma(d/2)\Gamma(b+1)\Gamma(d/2)\Gamma(b'+1)} \\ &\times \sum_{p,p'} \frac{(-1)^{p+p'} 2^{-b-b'} \Gamma(p-b)\Gamma(p-b')}{p!\Gamma(-b)p'!\Gamma(-b')} \\ &\times \int_0^{\infty} x^{d/2-1} e^{-2x} L_p^{(d/2-1)}(2x) L_{p'}^{(d/2-1)}(2x')\end{aligned}$$

With orthogonality relations for the Laguerre and then summing for p, we finally get

$$\|R\|^2 = \sum_a \sum_{b+b'=a} \frac{d_b(t) d_{b'}(t)}{\Gamma(b+1)\Gamma(b'+1)} \frac{\Gamma(d/2+a)}{2^{d/2+a}} \tag{24}$$

For given a, $2^{-a}[\Gamma(a+d/2)]/\Gamma(b+1)\Gamma(b'+1)$ is maximum when $b=b'=a/2$. If we rewrite the index set as $a(n) = (n+P_0)/\xi$, the above expression is bounded by $Kn^{(d-3)/2}$, where K is a constant depending on d and ξ. For $d \leqslant 3$, we get the simple majoration for $\|R\|^2$

$$\|R\|^2 < \sum_n \sum_m |d_m|\,|d_{n-m}| \text{ const}$$

or

$$\|R\| < \text{const} \sum_n |d_n(t)| \tag{25}$$

and it is enough to prove that $N(t) = \sum_n |d_n(t)|$ goes to zero. Let n_0 be the first index such that $d_{n_0}(0) \neq 0$.

For $n_0 \leqslant n \leqslant 2n_0 + P_0 - 1$ we have

$$|d_n(t)| \leqslant |d_n(0)|e^{-\gamma_n t} \leqslant |d_n(0)|e^{-\gamma_{n_0} t}$$

where γ_n is defined in (22a).

For $n \geqslant 2n_0 + P_0$, we get from (21)

$$|d_n(t)| < \exp(-\gamma_n t)\left[|d_n(0)| + \frac{1}{\beta_n}\int_0^t \exp(\gamma_n t') \sum_{m+m'=n-P_0} |d_m(t')|\,|d_{m'}(t')|\,dt' \right]$$

$$< \exp(-\gamma_{n_0} t)\left[|d_n(0)| + \frac{1}{\bar{\beta}}\int_0^t \exp(\gamma_{n_0} t') \sum_{m+m'=n-P_0} |d_m(t')|\,|d_{m'}(t')|\,dt' \right]$$

where $\bar{\beta} = \beta_{n_0} = (2n_0 + 2P_0 + \xi)/\xi$ or

$$N(t)\exp(\gamma_{n_0} t) < N(0) + \frac{1}{\bar{\beta}}\int_0^t \exp(\gamma_{n_0} t') N^2(t')\,dt'$$

or

$$N(t) < \frac{\bar{\beta}\gamma_{n_0} N(0)}{N(0) + \left[\bar{\beta}\gamma_{n_0} - N(0)\right]\exp(\gamma_{n_0} t)} \tag{26a}$$

whenever

$$N(0) < \gamma_{n_0}\bar{\beta}$$

i.e.,

$$N(0) < \left(\frac{n_0 + P_0 - \xi}{n_0 + P_0 + \xi}\right)\left(\frac{2n_0 + 2P_0 + \xi}{\xi}\right) \tag{26b}$$

Then, when $N(0)$ is small enough, the deviation from equilibrium goes to zero for long times. The proof holds for $d \leqslant 3$, which is the most interesting case, but a similar argument may be used for $d > 3$; the proof is given in Appendix D.

Result (26) has other interesting consequences. First a slight majoration of (26a) gives

$$N(t) < A\exp(-\gamma_{n_0} t) \qquad \text{where} \quad A = \frac{\bar{\beta}\gamma_{n_0} N(0)}{\bar{\beta}\gamma_{n_0} - N(0)}$$

and in (21), we replace largely $|\sum_{m+m'=n-P_0} d_m(t')d_{m'}(t')|$ by $N^2(t')$, whence

$$|d_n(t)| < \exp(-\gamma_n t)\left[|d_n(0)| + \frac{A^2}{\beta_n(2\gamma_{n_0} - \gamma_n)} \right] + \frac{A^2\exp(-2\gamma_{n_0} t)}{\beta_n(2\gamma_{n_0} - \gamma_n)}$$

then

$$\frac{|d_n(t)|}{|d_{n_0}(t)|} < C_1 \exp\left[-(\gamma_n - \gamma_{n_0})t\right] + C_2 \exp(-\gamma_{n_0} t) \tag{27a}$$

and $d_n(t)$ is negligible compared to $d_0(t)$ for long times. Then let

$$N_l = \sum_{n=n_0}^{2n_0+P_0-1} |d_n(t)| = \exp(-\gamma_{n_0} t) \sum_{n=n_0}^{2n_0+P_0-1} |a_n(0)| \exp\left[-(\gamma_n - \gamma_{n_0})t\right]$$

be the "linearized" part of $N(t)$ which includes only terms without quadratic terms. We have $N_l(t) > \exp(-\gamma_{n_0} t)|d_{n_0}(0)|$. The remaining (nonlinear) terms are

$$N_{nl} = \sum_{n \geqslant P_0+2n_0} |d_n(t)|$$

$$< \exp(-\gamma_{2n_0+P_0} t)\left[N_{nl}(0) + \frac{1}{\beta}\int_0^t \exp(\gamma_{2n_0+P_0} t') N^2(t')\, dt'\right]$$

$$< \exp(-\gamma_{2n_0+P_0} t)\left[N_{nl}(0) + \frac{A^2}{\bar{\beta}(2\gamma_{n_0} - \gamma_{2n_0+P_0})}\right] + \frac{A^2 \exp(-2\gamma_{n_0} t)}{\gamma_{2n_0+P_0} - 2\gamma_{n_0}}$$

and we have

$$\lim_{t\to\infty} \frac{N_{nl}(t)}{N_l(t)} = 0 \tag{27b}$$

and the large-time behavior is dominated by N_l. Let us notice that if in N_l we add $\sum_{n \geqslant P_0+2n_0} |d_n(0)| \exp(-\gamma_n t)$ and subtract these terms in N_{nl}, then property (27) still holds for this new ratio of nonlinear part versus linear one.

3.1.3. Positive Solutions. We indicate now, how we obtained examples of positive polynomially decreasing solutions. Positivity at positive times follows from positivity at $t = 0$.[6,8,16] It is enough to prove that there are infinitely many positive initial distributions $F(x, 0)$ so that inequality (26b) for $N(0)$ is fulfilled.

A simple class of solutions is the so-called fundamental positive solutions, i.e.,

$$d_{n_0}(0) \neq 0, \qquad d_n(0) = 0 \quad \forall n \neq n_0$$

As $\phi a_{n_0}(x)$ is first oscillating, then goes to infinity, it is easy to choose $d_{n_0}(0)$ small enough with the correct sign so that

$$1 + d_{n_0}(0) \frac{\Gamma(a_{n_0} + d/2)}{\Gamma(a_{n_0} + 1)\Gamma(d/2)} \phi\left(-a_{n_0}, \frac{d}{2}, x\right)$$

is positive for all x and inequality (26b) is true. When $t \neq 0$, other terms come in, but only for indices $a = (k+1)(P_0 + n_0)/N_0$.

Another way to get positive solutions when $a(n) = (n + P_0)/N_0$, N_0 integer, consists in choosing first a function which is positive, *known in closed form*, and may be expanded in Laguerre polynomials. Many examples were given in previous papers. We add to this positive function a unique hypergeometric $\phi(-(n_0 + P_0)/N_0, d/2, x)$ with $d_{n_0}(0)$ small enough and with the correct sign, so that the total function is positive. This "almost Laguerre" situation will be largely exploited for numerical purposes in Section 4. More generally we could add a *finite number* of hypergeometrics, where the coefficients are small enough with appropriate signs, in such a way that the whole $F(x,0)$ is positive.

3.2. Similarity Solutions

We start with system (16b)–(17b) and look for solutions in the form $\delta_q(t) = \bar{\delta}_q e^{-cqt}$ for a subset of values of q [otherwise $\delta_q(t) \equiv 0$]. We have $\delta_0(t) = 1$; let Z_0 be the first nonzero integer such that $\bar{\delta}_{Z_0} \neq 0$; necessarily $\delta_{Z_0}(t) = \bar{\delta}_{Z_0} \exp[-t(Z_0 - \xi)/(Z_0 + \xi)]$ and the unique possible relaxation constant c is

$$c = \frac{1}{Z_0} \frac{Z_0 - \xi}{Z_0 + \xi} \tag{28}$$

The differential system (17b) is replaced by a numerical system

$$(q - q_\alpha)(q_\beta - q)\bar{\delta}_q = q_\alpha q_\beta \sum_{\substack{m+m'=q \\ m,m' \geqslant 1}} \bar{\delta}_m \bar{\delta}_{m'} \tag{29}$$

where

$$q_\alpha = Z_0 \tag{30a}$$

is an integer,

$$q_\beta = \xi(Z_0 + \xi)/(Z_0 - \xi) \tag{30b}$$

and $q_\alpha, q_\beta > \xi > 0$. Conversely, q_α and $q_\beta > 0$ being given, ξ is the positive root of the algebraic equation

$$\xi^2 + \xi(q_\alpha + q_\beta) - q_\alpha q_\beta = 0 \tag{31}$$

and $0 < \xi < q_\alpha, q_\beta$. Then, the similarity solution reads in the energy variable

$$F(x,t) = \exp(-x)\left[1 + \sum_q \bar{\delta}_q \exp\left(-\frac{\xi}{q_\alpha q_\beta} qt\right)\phi\left(-\frac{q}{\xi}, \frac{d}{2}, x\right)\right] \tag{32}$$

Two situations may occur:

(i) q_α is an integer and q_β is real positive, but *not* an integer (nondegenerate case). Then, the sequence $\{\bar{\delta}_q\}$ depends on one parameter only, $\bar{\delta}_{q_\alpha}$ and the indices of the nonzero $\bar{\delta}_q$ are multiples of q_α: $2q_\alpha, 3q_\alpha, \ldots$. Setting $\bar{\delta}_{kq_\alpha} = \bar{\bar{\delta}}_k$, we get the simplified equation

$$(k-1)\left(\frac{q_\beta}{q_\alpha} - k\right)\bar{\bar{\delta}}_k = \frac{q_\beta}{q_\alpha} \sum_{\substack{m+m'=k \\ m,m' \geqslant 1}} \bar{\bar{\delta}}_m \bar{\bar{\delta}}_{m'}$$

Notice that such a situation is very general. Given q_α integer and $q_\beta \neq q_\alpha$, q_β not an integer, there exists always $\xi < q_\alpha, q_\beta$ and $c > 0$, ξ being integer, rational or not rational [Eqs. (31) and (28)]. Similarly, given q_α and ξ, it is almost always (i.e., except for a countable number of values of ξ) possible to find q_β not an integer. Nevertheless, the Wu solution does not enter this category as in the pure Laguerre case we have $\xi = 1$, $q_\alpha = Z_0 = 2$ (in general), and $q_\beta = 3$. It is remarkable that this family of functions which was long the only one to be known, is in some sense marginal.

(ii) q_α and q_β are both positive integers (degenerate case), and therefore the solutions depend on two parameters $\bar{\delta}_{q_\alpha}$ and $\bar{\delta}_{q_\beta}$; more indices q appear (in the Wu solutions, all indices $q \geqslant 2$ appear).

Even if this possibility is more rare than case (i), there are still infinitely many solutions, namely, q_α and q_β being given integers, there always exist ξ solutions of (31), such that $q_\alpha, q_\beta < \xi$ and $c > 0$ is given by (28); and conversely, given q_α there exists an infinite (countable) set of ξ such that q_β is an integer. In particular, when ξ is itself an integer N_0 ($\xi = N_0$), it is always possible to get at least one similarity solution. From (31), we look for couples of integers (q_α, q_β) such that

$$(q_\alpha - N_0)(q_\beta - N_0) = 2N_0^2 \tag{33}$$

For every N_0, we have the three couples of solutions $(N_0 + 1, N_0 + 2N_0^2)$ $(N_0 + 2, N_0 + N_0^2)$ and $(2N_0, 3N_0)$; they are not all distinct for $N_0 = 1, 2$ and they are the only possible solutions when N_0 is a prime integer. For the couple $(2N_0, 3N_0)$, the nonzero $\bar{\delta}_q$ correspond to indices multiple of $N_0 (q = kN_0)$ and the associated eigensolutions are the Laguerre functions $L_k e^{-x}$, beginning with $L_2 e^{-x}$ and $L_3 e^{-x}$, i.e., the Bobylev–Krook–Wu solution again! More generally, if $(q_{\alpha_1}, q_{\beta_1})$ is a couple of solutions of (33) for some integer $N_{0,1}$, then $(mq_{\alpha_1}, mq_{\beta_1})$ is a solution for $N_{0,2} = mN_{0,1}$. As an example, we give below the solutions for $N_0 = 1, \ldots, 4$.

(i) $N_0 = 1$ (pure Laguerre case) the three solutions reduce to the unique couple (2, 3). They were first indicated by Bobylev. A subclass is the one-parameter family found by Krook, Wu, and Bobylev:

$$F(x,t) = (1-z)^{-5/2}\left(1 - \frac{5}{2}z + \frac{xz}{1-z}\right)\exp\left(\frac{xz}{z-1}\right), \qquad 0 < z < 1$$

corresponding to[7] $\delta_q(t) = (-1)^{q+1}(ze^{-t/6})^q(q-1)$. The whole family was studied in Ref. 6 but it was impossible to find any positive solution except that of Krook–Wu.

(ii) $N_0 = 2$ [eigenfunctions $e^{-x}\phi(-n/2, d/2, x)$] we get two distinct couples (3, 10) and (4, 6); the second one is the (2, 3) solution for $N_0 = 1$.

(iii) $N_0 = 3$, the eigenfunctions are $e^{-x}\phi(-n/3, d/2, x)$; we have three distinct solutions (4, 21), (5, 12), and (6, 9), which is again equivalent to (2, 3) for $N_0 = 1$.

(iv) $N_0 = 4$ the eigenfunctions are $e^{-x}\phi(-n/4, d/2, x)$; there are three solutions: (5, 36), (6, 20), which is the same as (3, 10) for $N_0 = 2$, and (8, 12), which is the same as (2, 3) for $N_0 = 1$.

Practically, it is very difficult to get an explicit closed form for these functions. Function $(p+1)G(p,t)$ is a series in variable $z = e^{-ct}p/(p+1)$, but Eq. (29) is equivalent to a nonlinear differential equation in z of the second order that we cannot handle easily except for the Krook–Wu family. As a consequence, it is difficult to study their behavior and, mainly, their positivity. All the examples that we studied numerically led to nonacceptable solutions and it is not a trivial problem to prove positivity without starting with set $\{d_n(0)\}$.

4. NUMERICAL CALCULATIONS

Though the mathematical results above are very similar to the pure Laguerre case, numerical effects are sometimes drastically modified because of the powerlike decrease of the new eigenfunctions. We were interested in two main points. The first one is related to the Tjon effect,[14] which was widely represented in the ordinary (Laguerre) situation[8] when the coefficient of the first Laguerre—i.e., $L_2^{(d/2-1)}$—is positive[10] and not too small[8] compared with the coefficient of the next Laguerre $L_3^{(d/2-1)}$. In that case, an overpopulation occurs at higher and higher energy as time goes by; the process is sketched in Fig. 1 and the effect can be seen either at fixed t (Fig. 1a) or at fixed x (Fig. 1b). It partially disappears when we add a true hypergeometric, and can be observed only for the intermediate times on the curves at fixed t or only for intermediate energies for those at fixed x.

The second study deals with the relaxation rate to equilibrium, which, in the Laguerre case, is bounded from below by 1/3. Let us define the reduced distribution function

$$\tilde{F}_R(x,t) = F(x,t)/F(x,\infty) = e^x F(x,t) \tag{34}$$

In the Laguerre case, it goes to 1 at least like $e^{-t/3}$ but cannot decrease

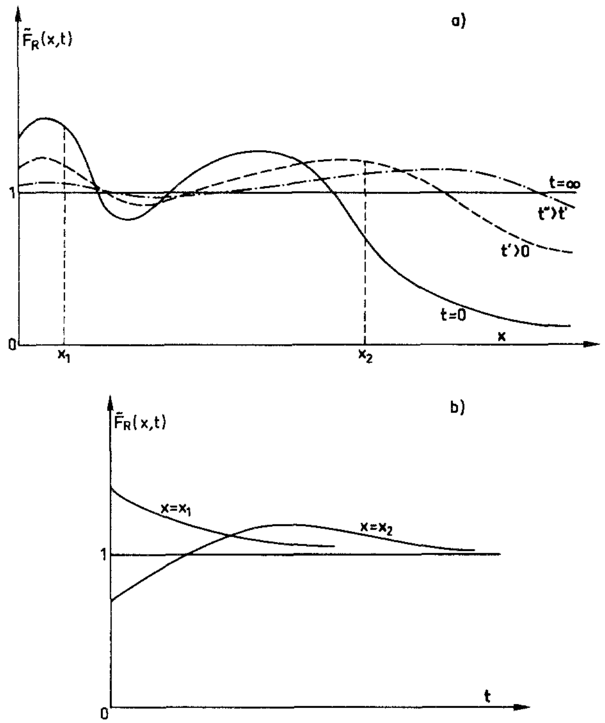

Fig. 1. Schematic evolution of the reduced distribution $\tilde{F}_R(x, t)$ for Tjon's effect. The initial $\tilde{F}_R(x, 0)$ must have two bumps and the maximum and zero of the last one moves toward the right as time goes by (Fig. 1a). In Fig. (1b) is plotted $\tilde{F}_R(x, t)$ for several x as a function of t; for large x $(= x_2)$, $F_R(x, t)$ is first below 1, then crosses the equilibrium value, goes to a maximum, then goes back uniformly to 1.

more slowly than $e^{-t/3}$. With the new eigenfunctions ψ_a, the relaxation mode is still determined by the first term in the expansion, which can become arbitrarily small. For example, the relaxation to equilibrium is $\exp[-t/(2N_0 + 1)]$ and $\exp[-(\eta - 1)/(\eta + 1)t]$ $(\eta > 1)$ in families $a(n) = (N_0 + 1 + n)/N_0$ and $a(n) = (n + 1)\eta$, which are special cases of (16a), and it becomes very slow when N_0^{-1} and $\eta - 1$ are nearly zero. As will be explained below, the linearized expression for $\tilde{F}_R$ is good enough in order to explain these effects.

These effects may be expected even if we are "not too far" from a pure Laguerre situation. Furthermore, if at $t = 0$, hypergeometric components are present, then necessarily, for x sufficiently large, $\tilde{F}_R$ will exceed value 1. If we want to observe when t increases an overpopulation effect, it will be convenient to introduce in supplement, at $t = 0$, a family of Laguerre terms.

So, we restrict ourselves to families

$$a(n) = (N_0 + 1 + n)/N_0 \tag{35}$$

N_0 integer, which involve Laguerre eigenfunctions $\psi_{kN} = e^{-x}L_k^{(d/2-1)}(x)$ as a subset, and true hypergeometrics $\psi_n \sim e^{-x}\phi[-(N_0 + 1 + n)/N_0, d/2, x]$ occur when $n + 1$ is not a multiple of N_0. We studied more precisely the cases $N_0 = 2[a(n) = (n + 3)/2]$ and $N_0 = 4[a(n) = (n + 5)/4]$ both for the description of the Tjon effect and of the relaxation rate which should be 1/5 and 1/9, respectively; we compared it to the Laguerre case $N_0 = 1$, already studied in previous publications.[6-8] The method consists again in integrating recursively the system (17a) up to large enough order. We did not get for small time the same precision as in the Laguerre case, so most of our results are plotted for $t > 10$. The reason is that integrating up to the 15th Laguerre polynomial is not too complicated for $N_0 = 1$, but is equivalent to reach to the 30th and 60th differential equation if $N_0 = 2$ and 4; so we limited ourselves to the 9th and 8th polynomials in our two examples. For $N_0 = 20$, the same precision (up to L_9) would imply the integration up to the 180th equation!

4.1. General Argument

Let us explain first how the introduction of true hypergeometrics $e^{-x}\phi(-a, d/2, x)$ is so important. For not too large energies x, $\phi_a = \phi(-a, d/2, x)$ behaves like L_{n+1} (n is the integer part of a). It has $(n + 1)$ positive zeros and oscillates between them; the qualitative aspect at small or intermediate energies is not basically changed. But for large x, ϕ increases exponentially, precisely $\phi_a \sim e^x x^{-a-d/2}$; then ϕ_a is negligible compared to ϕ_b whenever $a > b$, in opposition to the Laguerre case. Moreover, as $d_a(t)$ decreases (in general) faster than $d_b(t)$, higher-order ϕ's play no role in asymptotic (large x or large t) behavior. Most of the information, then, comes from the respective location for the $d_n(0)$'s of both the Laguerre part and the first true hypergeometric. Concerning the norm of $F - 1$ it was previously shown that it decreases like the time dependence of the first $d_n(t) \neq 0$. This general feature will be slightly modulated here for $\tilde{F}_R(x, t)$ because another variable, the energy x, is present.

If the index of the first term in the expansion of $\tilde{F}_R - 1$ is a true hypergeometric term then its contribution $d_n(t)\psi_n(x)$ is dominant both at large t and large x. If on the contrary the expansion begins with Laguerre term, then the contribution $d_n(t)\psi_n(x)$ of the first true hypergeometric is dominant only for large x at small t and the rough behavior is that of the Laguerre part. More precisely starting with $\tilde{F}_R(x, t)$ which expands for family (35) as

$$\tilde{F}_R(x,t) = 1 + \sum_{n \geqslant n_0} d_n(t)\phi\left(-\left(\frac{N_0+1+n}{N_0}\right), \frac{d}{2}, x\right) \tag{36}$$

two cases occur. In the first case $(n_0+1)/N_0$ is not an integer and n_0 corresponds to a true hypergeometric term, whereas in the second case it is an integer and n_0 corresponds to a Laguerre.

(i) In the first case for x sufficiently large we find

$$\tilde{F}_R(x,t) \simeq 1 + \text{const}\exp\left[-t\left(\frac{n_0+1}{n_0+1+2N_0}\right)\right]\exp(x)$$
$$x^{-[d/2+1+(n_0+1)/N_0]} \tag{37}$$

where we have approximated the hypergeometric $\phi(-1-(n_0+1)/N_0, d/2, x)$ by its asymptotic behavior. When t increases $(\tilde{F}_R - 1)$, which is positive, decreases monotonously to zero. No Tjon effect may be observed when (37) becomes valid. The relaxation rate is $(n_0+1)/(n_0+1+2N_0)$ and the process to equilibrium with $N_0 > 1$ can be slower than $1/3$ if $n_0 < N_0 - 1$. For instance, if $N_0 = 2$ we can have $n_0 = 0$, if $N = 3$, $n_0 = 0$ and 1, if $N = 4$, $n_0 = 0, 1, 2$, and so on. On the other hand, for intermediate x values, (37) is not valid, we must include the Laguerre part in the linear approximation, and the Tjon effect can be present. Finally let us remark that if $n_0 = 0$, the relaxation rate is $1/(2N_0+1)$ and *can be as small as we want choosing sufficiently large N_0 values.*

(ii) In the second case n_0 is equal either to $N_0 - 1$, or $2N_0 - 1, \ldots,$ or more generally to $kN_0 - 1$. In that case the first term in the expansion corresponds either to $L_2^{(d/2-1)}(x)$ or $L_3^{(d/2-1)}(x) \ldots$. Let $n_1 > n_0$ be the first index corresponding to a true hypergeometric. In order to simplify the discussion we consider $n_0 = N_0 - 1$ and retain in the linear expansion only the terms corresponding to n_0 and n_1:

$$\tilde{F}_R(x,t) - 1 \simeq \text{const}_1 \exp(-t/3)\left[x^2 + \text{const}_2 \exp(x - \gamma t)x^{-(d/2+\mu)}\right] \tag{38}$$

where $\gamma = (n_1+1)/(n_1+1+2n_0) - 1/3 > 0$ and $\mu = (n_1+1+N_0)/N_0$. There exist two different regimes:

If $x \gg \gamma t$, which can always be satisfied for small t values and x sufficiently large, then the second term in (38) is dominant. Consequently the relaxation is governed by the true hypergeometric term.

If $x \ll \gamma t$, which can always be satisfied for any fixed x value and t sufficiently large, then the first term in (38) is the dominant one and the relaxation is dominated by the Laguerre contribution $(1/3)$. For the transition regime where $x \simeq \gamma t$, it is clear that we must include other Laguerre terms between n_0 and n_1.

4.2. Modified Tjon Effect

As we already explained, we restricted ourselves to families (35), with $N_0 = 2, 4$ and $\tilde{F}_R(x, 0)$ is "not far" from a pure Laguerre situation where Tjon's effect was already observed.[8] The perturbation is the unique term $\phi((n_0 + 1)/N_0, d/2, x)$ with a small coefficient $d_0(0)$ (say, 10 or 100 times smaller than the first Laguerre coefficients) so that it dominates only at high energies ($x > 15$) and the function increases abruptly. For intermediate x ($x < 12$) it looks like the ordinary Laguerre case with two bumps. For small t ($t < 3$) and not too large energies ($x \sim 10$), we notice a displacement of the intermediate bump (Figs. 2 and 3). For large t, it disappears, as the hypergeometric is again predominant. As in previous works, we have plotted $\tilde{F}_R(x, t)$ as a function of t for several x (Figs. 4 and 5). When x is in the well $[\tilde{F}_R(x, 0) < 1]$, $\tilde{F}_R(x, t)$ first increases, becomes greater than 1, then decreases to 1 monotonously like in Tjon's observation.

We have done the same study with $N_0 = 2$, but $n_0 = 1$ and the first

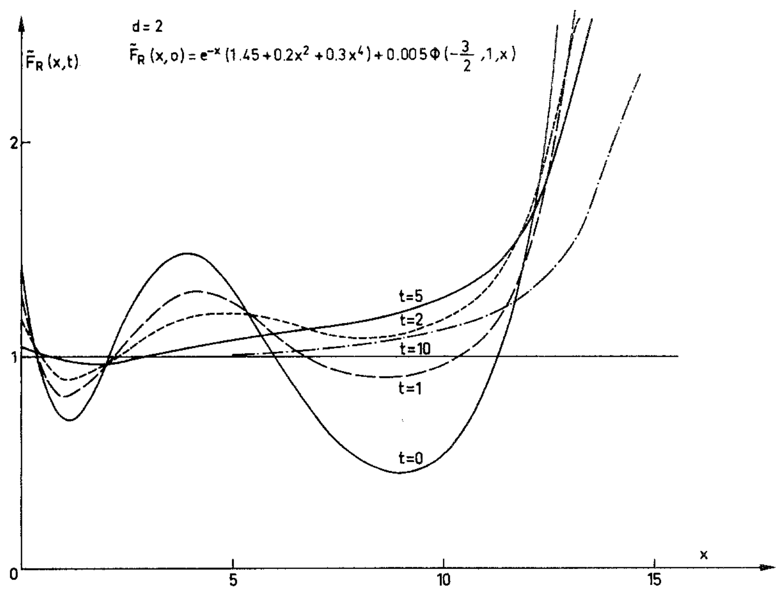

Fig. 2. Evolution in time of the reduced distribution function $\tilde{F}_R(x, t)$ for a two-dimensional gas in an "almost" Laguerre situation with Tjon effect.

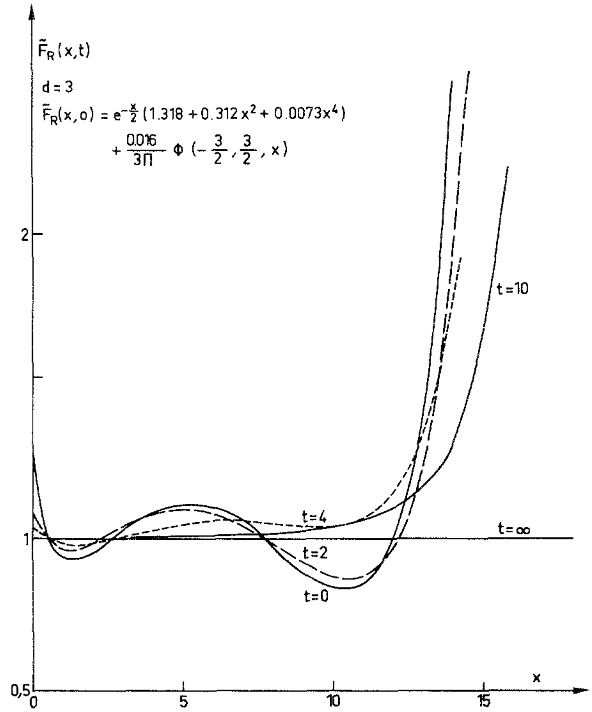

Fig. 3. Same as Fig. 2 for a three-dimensional gas.

hypergeometric occurs for $d_4(0)$ (i.e., $n_1 = 4$). Tjon's effect is also effective both on finite energy and time intervals; practically, for the time and energy we can numerically reach with good precision, the relaxation curves are not very different from that presented above.

4.3. Relaxation to Equilibrium

In the pure Laguerre situation the reduced distribution function $\tilde{F}_R(x,t)$ goes to one exponentially like $e^{-t/3}$ and it was generally assumed that it is the slowest mode possible. Now, in an enlarged family ($N_0 > 1$) with $a(n) = (n + 1 + N_0)/N_0$, when $d_0(0) \neq 0$ ($n_0 = 0$), the relaxation rate for x large should be $1/(2N_0 + 1)$ even if $d_0(0)$ is small. For intermediate energies, this is not so clear as Laguerre part is not negligible and we had to test it numerically.

These results have been checked for $N_0 = 1$ (pure Laguerre), 2 and 4, denoted as I, II, and III. In Figs. 6–8, we have plotted the instantaneous

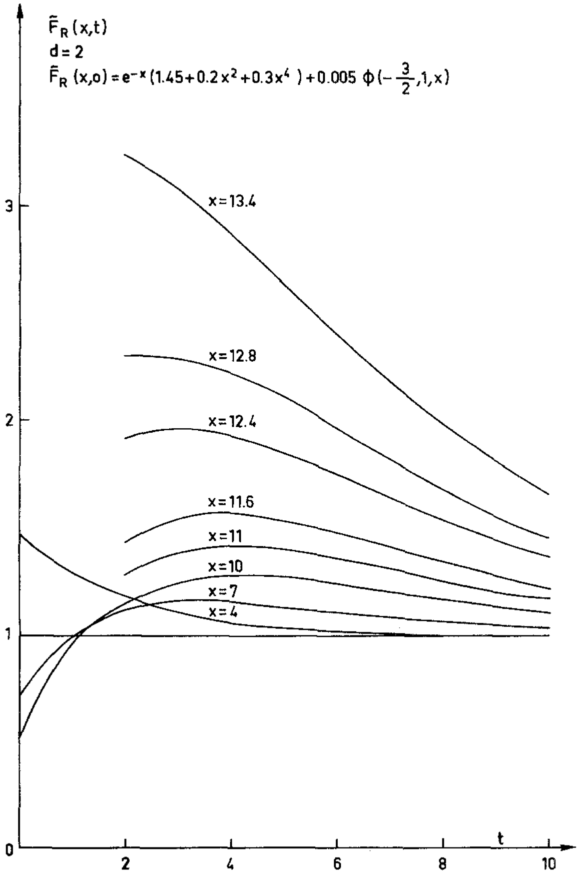

Fig. 4. Plot of $\tilde{F}_R(x,t)$ as a function of t for the two-dimensional gas and several energies. Tjon's behavior corresponds to energies 7–11. The transition to the pure linear regime occurs near $x \simeq 13$. We have not plotted $\tilde{F}_R(x,t)$ for small t values when the precision was not sufficient.

relaxation rate

$$\gamma(x,t)=\frac{1}{\Delta t}\log\left[\frac{\tilde{F}_R(x,t)-1}{\tilde{F}_R(x,t+\Delta t)-1}\right]$$

as a function of time t for several values of x and the three families. The initial reduced distributions are again not far from a pure Laguerre situation, with a small perturbation $d_0(0)=d_0$. Depending on the family, $\gamma(x,t)$ goes fairly well to the respective theoretical values 1/3, 1/5, and 1/9; for large x ($x>15$), the limit is already reached at $t\simeq 10$ for families II and III because of the predominance of ϕ_0. The process is slower for small x,

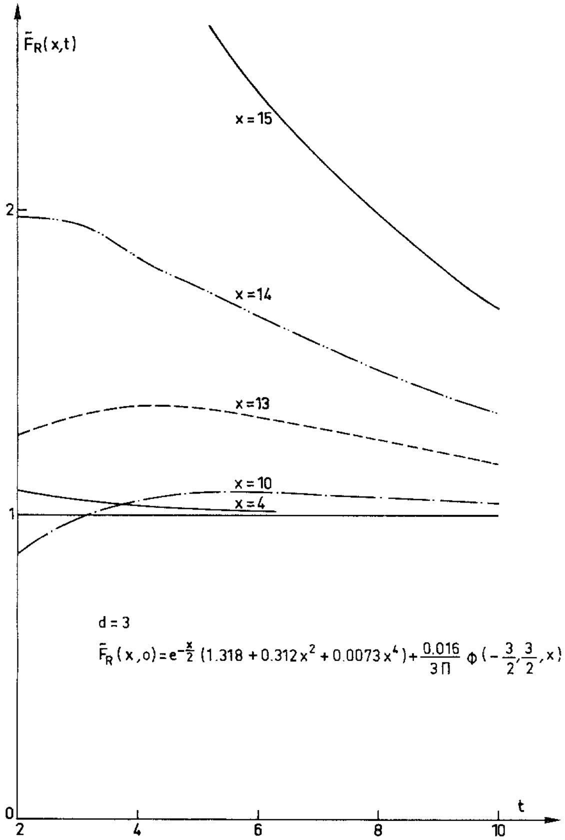

Fig. 5. Same as Fig. 4 for a three-dimensional gas. The transition to pure linear regime occurs at $x \simeq 14$.

mainly for energies which are in the well at $t = 0$ ($x \sim 10$), but at $t = 20$, $\gamma(x,t)$ is already far lower than the pure Laguerre limit 1/3. In Figs. 9a, b we have plotted $\gamma(x,t)$ for a given x and different values of d_0 for families II and III, respectively: the convergence to γ_0 improves when d_0 increases, as the domination of ϕ_0 takes place earlier.

In Fig. 10 we have envisaged the case where the first true hypergeometric is not the first term of the expansion of $\tilde{F}_R(x,0)$. We limited ourselves to $N_0 = 2$, $d = 2$, and the hypergeometric is $\phi(-7/2, 1, x)$ coming after $L_2^{(0)}(x)$ and $L_3^{(0)}(x)$ ($n_0 = 1$, $n_1 = 4$). When x is in the well ($x < 20$), the limit 1/3 is rapidly reached as in the pure Laguerre. For large x ($x > 30$) and small times, the Laguerre is negligible and the relaxation rate is $\gamma(x,t) \sim 5/9$; for example, for $x \sim 40$, $\gamma(x,t)$ is undistinguishable from 5/9, up to values of times of the order of 45. When t increases, $\gamma(x,t)$ slows

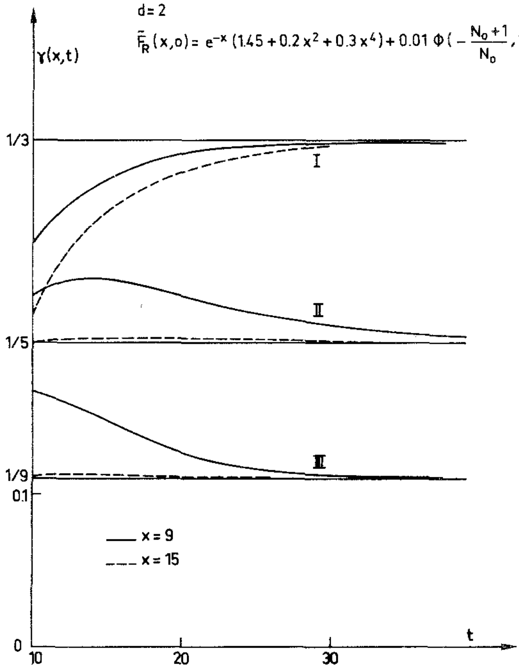

Fig. 6. Plot of $\gamma(x, t)$ as a function of t for several x and $d = 2$. Family I, pure Laguerre case (no ϕ function); family II, $N_0 = 2$; family III, $N_0 = 4$.

down to the true limit $1/3$ and the transition occurs for a time t_x, increasing with x. No precise study of this transition time has been done.

ACKNOWLEDGMENTS

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APPENDIX A

1. Derivation of the N.L.P.D.E. for G for the $3d$ Case

We start from the Boltzmann equation (5),

$$\partial_t f + f = N \int d\mathbf{w} \int \frac{d\Omega}{4\pi}\, f(v', t) f(w', t) \tag{A1}$$

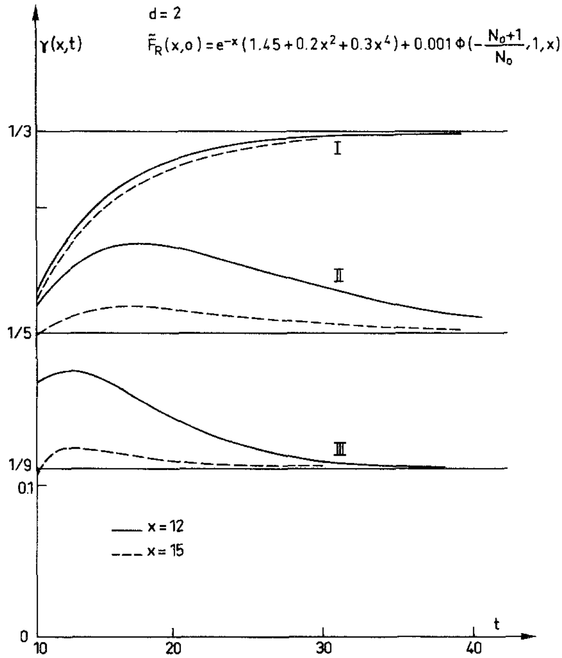

Fig. 7. The same as Fig. 6 with other values of the perturbation parameter ($d_0 = 0$, $d_0 = 0.001$, $d_0 = 0.001$).

where $N = \int e^{-w^2} d\mathbf{w} = \pi^{-3/2}$; $\mathbf{v}'$ and $\mathbf{w}'$ are the velocities after collision:

$$v'^2(w'^2) = v^2 \frac{1 \pm \cos x}{2} + w^2 \frac{1 \mp \cos x}{2} \pm vw \sin x \sin\theta \cos\epsilon$$

$\Omega = (x, \epsilon)$ are standard notations for the diffusion direction angles and $\theta = (\mathbf{v}, \mathbf{w})$. Using transformation (1a), we get

$$\partial_t G + G = \frac{1}{\pi^3} \int \int d\mathbf{v}\, d\mathbf{w} \int \frac{d\Omega}{4\pi} f(v', t) f(w', t) \phi\left(1, \tfrac{3}{2}, -pv^2\right)$$

or, after exchanging variable $\mathbf{v}'$, $\mathbf{w}'$ and $\mathbf{v}$, $\mathbf{w}$,

$$\begin{aligned} \partial_t G + G &= \frac{1}{\pi^3} \int \int d\mathbf{v}\, d\mathbf{w}\, f(v, t) f(w, t) \int \frac{d\Omega}{4\pi} \phi\left(1, \tfrac{3}{2}, -pv'^2\right) \\ &= \frac{2}{\pi^2} \int_0^\infty v^2 f(v, t) \int_0^\infty w^2 f(w, t) I(p, v, w) \end{aligned} \tag{A2}$$

$$\begin{aligned} I(p, v, w) &= \int_0^\pi \sin x\, dx \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\epsilon\, \phi\left(1, \tfrac{3}{2}, -pv'^2\right) \\ &= \int_0^\pi \sin x\, dx \sum_{n \geqslant 0} \lambda_n (-p)^n \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\epsilon\, v'^{2n} \end{aligned}$$

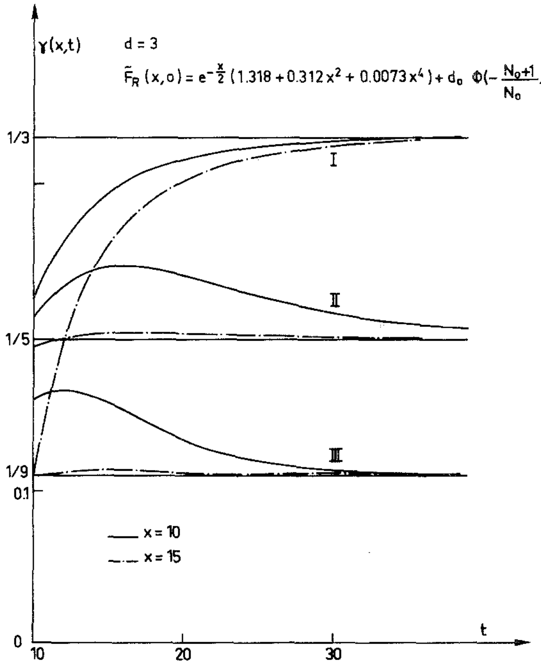

Fig. 8. The same as Fig. 6 for a $3d$ gas. Family I, $d_0 = 0$; family II, $d_0 = 0.016/\pi$, $\phi_0 = \phi(-3/2, 3/2, x)$; family III, $d_0 = \Gamma(11/4)/\Gamma(9/4)\Gamma(3/2)$, $\phi_0 = \phi(-5/4, 3/2, x)$.

with $\lambda_n = \Gamma(3/2)/\Gamma(n+3/2)$. The last two integrals may be performed from Krook–Wu techniques[2] for calculating $\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\epsilon\, e^{-pv'^{2n}}$ and we get

$$(n+1)\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\epsilon\, v'^{2n} = \frac{2\pi}{vw \sin x}\left[\left(v\cos\frac{x}{2} + w\sin\frac{x}{2}\right)^{2n+2} - \left(v\cos\frac{x}{2} - w\sin\frac{x}{2}\right)^{2n+2}\right]$$

The integration over the diffusion angle χ is too classical. We expand the polynomials and finally get for I,

$$I = 4\pi \sum_{n\geqslant 0} (-p)^n \lambda_n \frac{\Gamma(n+3/2)\Gamma(1/2)}{n+1} \sum_{m=0}^{n} \frac{v^{2m}}{\Gamma(m+3/2)} \frac{w^{2n-2m}}{\Gamma(n-m+3/2)} \tag{A3}$$

Variables v and w can be separated by replacing

$$\frac{\lambda_n \Gamma(n+3/2)}{n+1}$$

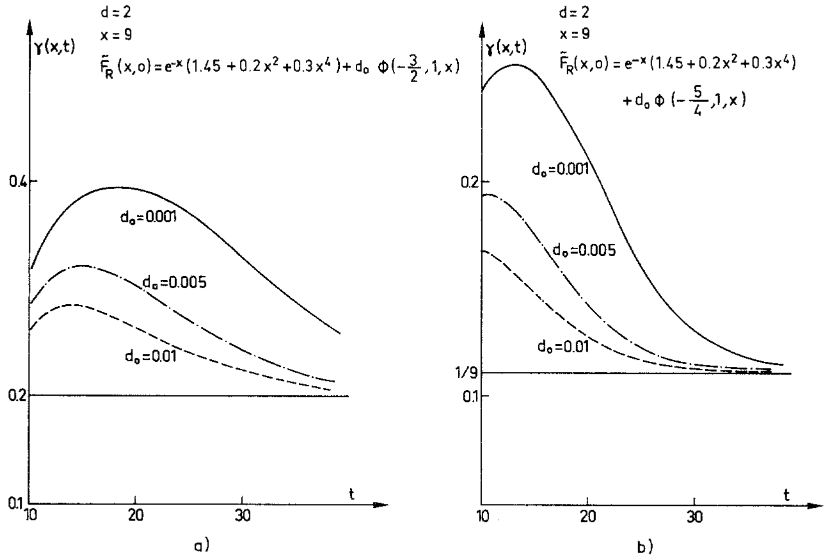

Fig. 9. Plot of $\gamma(x, t)$ as a function of t for $x = 9$ and dimension 2. (a) For family II and different values of d_0, (b) for family III.

by the integral

$$\Gamma\left(\frac{3}{2}\right)\int_0^1 u^n\, du$$

whence, setting $m' = n - m$,

$$I = 4\pi\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\sum_{m,m'}(-p)^{m+m'}\frac{v^{2m}}{\Gamma(m+3/2)}\,\frac{w^{2m'}}{\Gamma(m'+3/2)}\int_0^1 u^{m+m'}\,du$$

$$= 8\pi\int_0^1 du\,\phi\left(1,\tfrac{3}{2}, -upv^2\right)\phi\left(1,\tfrac{3}{2}, -upw^2\right) \tag{A4}$$

then we rewrite (A2) as

$$\partial_t G + G = \frac{16}{\pi}\int_0^1 du\int_0^\infty dv\, v^2 f(v,t)\phi\left(1,\tfrac{3}{2}, -upv^2\right)$$

$$\times\int_0^\infty dw\, w^2 f(w,t)\phi\left(1,\tfrac{3}{2}, -upw^2\right)$$

$$= \int_0^1 du\, G^2(up,t) = \frac{1}{p}\int_0^p du\, G^2(u,t)$$

and by derivation $(\partial/\partial p)(\partial_t + 1)pG = G^2$.

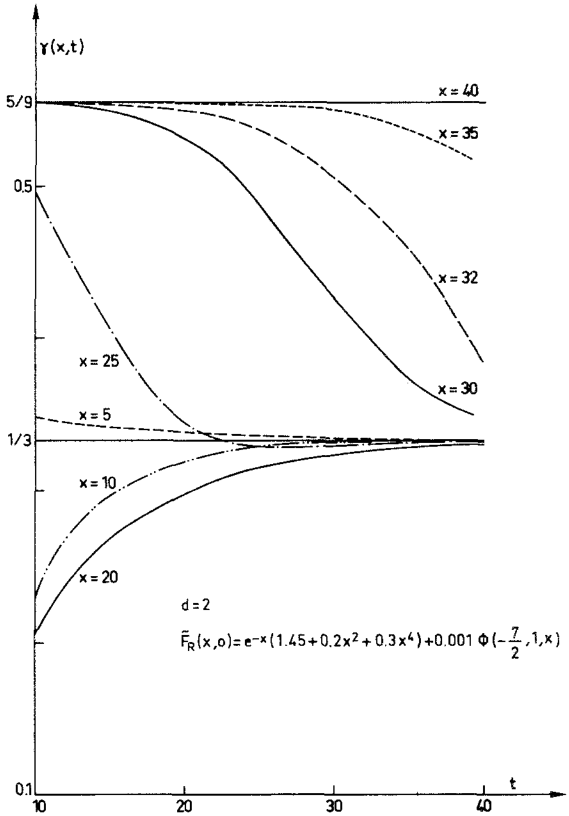

Fig. 10. Plot of $\gamma(x, t)$ as a function of t for family II and different values of x when the first term in the expansion is a Laguerre polynomial.

2. Proof for $d \geqslant 3$

Equation (A1) is replaced by

$$\partial_t f + f = N_d \int d^d \mathbf{w} \int \frac{d\Omega_d}{S_d} f(v', t) f(w', t) g_d(x) \tag{A5}$$

where $N_d = \pi^{-d/2}$, Ω_d and S_d are the d-dimensional solid angle and unit surface, respectively:

$$d\Omega_d = (\sin x)^{d-2} (\sin \epsilon)^{d-3} (\sin \epsilon_1)^{d-4} \cdots \sin \epsilon_{d-4}\, dx\, d\epsilon\, d\epsilon_1 \cdots d\epsilon_{d-3}$$

and the cross section[8]

$$g_d(x) = (\sin x)^{3-d} \frac{\Gamma^2[(d-1)/2]}{\Gamma^2(d/2)} \frac{\pi}{4}$$

After integrating over angles $\epsilon_1 \ldots \epsilon_{d-3}$, $0 < \epsilon_1, \epsilon_2, \ldots, \epsilon_{d-4} < \pi$, $0 < \epsilon_{d-3} < 2\pi$ and using transformation (1a), one gets

$$\partial_t G + G = \frac{1}{2} \frac{\pi^{-1/2}\Gamma[(d-1)/2]}{\Gamma^2(d/2)\Gamma(d/2-1)} \int_0^\infty v^{d-1}\,dv \int_0^\infty w^{d-1}\,dw \times f(v,t) f(w,t) I_d(p,v,w)$$

where

$$I_d(p,v,w) = 2\int_0^\pi \sin x\,dx \int_0^\pi \sin\theta\,d\theta \int_0^\pi d\epsilon (\sin\epsilon)^{d-3} \phi\left(1, \frac{d}{2}, -pv'^2\right)$$

reduces to $I(p,v,w)$ if $d = 3$.

Calculations for every dimension were already performed in Ref. 8. We get for

$$\langle v'^{2n} \rangle = \int_0^\pi \sin x\,dx \int_0^\pi \sin\theta\,d\theta \int_0^\pi v'^{2n} (\sin\epsilon)^{d-3}\,d\epsilon$$

the final value

$$\frac{4\Gamma(1/2)\Gamma(d/2-1)}{\Gamma[(d-1)/2]} \sum_{m+m'=n} \int_0^1 du(-pu)^{m+m'} \frac{v^{2m}\Gamma(d/2)}{\Gamma(m+d/2)} \frac{w^{2m'}\Gamma(d/2)}{\Gamma(m'+d/2)}$$

and the proof ends like for $d = 3$. ■

3. Remark

Knowing the eigenfunctions (12) of the linearized system it is possible to obtain directly system (15) when $d = 2$ and, in a more complicated way, when $d = 3$.

3.1. 2*d* Gas. Starting with the Krook–Wu equation (4) and using expansion (13), we rewrite the B.E. as

$$\sum_a \left[\dot{d}_a(t) + d_a(t)\right] e^{-x} \phi(-a,1,x) = \sum_{b,b'} d_b(t) d_{b'}(t) \int_x^\infty \frac{e^{-x'}}{x'}\,dx' \int_0^{x'} dx'' \phi(-b,1,x'-x'') \phi(-b',1,x'')$$

Using the convolution properties of the hypergeometric the integral over x'' becomes[17]

$$x'\phi(-b-b',2,x')$$

Using

$$e^{-x'}\phi(-b-b',2,x') = -\frac{1}{b+b'+1}\frac{d}{dx'}e^{-x'}\phi(-b-b',1,x')$$

[Ref. 17, p. 255, formula (13)], we perform the integration over x' and system (15) follows.

3.2. 3*d* Gas. We use the Laplace transform $\tilde{F}(p,t)$ of $F(p,t)$, which expands as

$$\tilde{F}(p,t) = 2\pi \sum_{\substack{a>1\\a=0}} d_a(t)\frac{1}{p^{3/2}}\left(\frac{p}{p+1}\right)^a \frac{\Gamma(a+3/2)}{\Gamma(a+1)}$$

Equation (A1) becomes

$$\sum_a (d_a + \dot{d}_a)\frac{\Gamma(a+3/2)}{\Gamma(a+1)}\frac{1}{p^{3/2}}\left(\frac{p}{p+1}\right)^a$$
$$= 2\pi N \sum_{b,b'} \frac{d_b d_{b'}}{\Gamma^2(3/2)}\frac{\Gamma(b+3/2)}{\Gamma(b+1)}\frac{\Gamma(b'+3/2)}{\Gamma(b'+1)}$$
$$\times \int_0^\infty dv\, v^2 e^{-v^2}\phi\left(-b,\tfrac{3}{2},v^2\right)\int_0^\infty dw\, w^2 e^{-w^2}\phi\left(-b',\tfrac{3}{2},w^2\right)J(p,v,w) \tag{A6}$$

where

$$J(p,v,w) = \int_0^\pi \sin x\, dx \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\epsilon\, e^{-pv'^2}$$

is the analog of $I(p,v,w)$ in (A2) and may be calculated in the same way, the coefficient λ_n is now $\lambda_n = 1/n!$. Expression (A4) for I becomes

$$J(p,v,w) = \frac{2}{\Gamma^2(3/2)}\int_0^1 du\, u^{1/2}(1-u)^{-1/2}\phi\left(1,\tfrac{3}{2},-puv^2\right)\phi\left(1,\tfrac{3}{2},-puw^2\right)$$

From Ref. 17, p. 287, formula 22, the entire right-hand side of (A6) may be rewritten as

$$\pi N \sum_{b,b'} d_b d_{b'} \int_0^1 du\, u^{1/2}(1-u)^{-1/2}(pu)^{b+b'}(1+pu)^{-b-b'-2}$$

The integral becomes (Ref. 17, p. 10, formula 12)

$$p^{b+b'}(1+p)^{-b-b'-3/2}\frac{\Gamma(b+b'+3/2)\Gamma(1/2)}{\Gamma(b+b'+2)}$$

whence the result.

APPENDIX B

From G and Eq. (3) let us define a new function

$$h\left(u=\ln\frac{p}{p+1},t\right)=-1+(p+1)G(p,t)$$

which satisfies a symmetric N.L.P.D.E.

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+\frac{\partial^2}{\partial t\,\partial u}-1\right)h(u,t)=h^2(u,t) \tag{B1}$$

First we consider the linear part (left-hand side) of (B1) and we see that the eigensolutions $e^{-\lambda t}e^{a_\lambda u}$ with $a_\lambda=(\lambda+1)/(1-\lambda)$ can be chosen exponential either in the t or u variables. The physical requirements for the moments M_0, M_1, and the behavior when $t\to\infty$ restrict to $0<\lambda<1$ and $a>1$.

Secondly for the whole nonlinear Eq. (B1) we can consider either an expansion like $\sum_a e^{au}\delta_a(t)$ which leads to the system (15) (studied in this paper) or another expansion $\sum_\lambda e^{-\lambda t}\delta_\lambda(u)$ where the coefficients δ_λ satisfy

$$\frac{d}{du}\delta_\lambda(1-\lambda)-\delta_\lambda(1+\lambda)=\sum_{\mu+\mu'=\lambda}\delta_\mu\delta_{\mu'} \tag{B2}$$

APPENDIX C

We start with expression (21) for $d_n(t)$

$$d_n(t)=e^{-\gamma_n t}\left[d_n(0)+\frac{1}{\beta_n}\int_0^t dt'\,e^{\gamma_n t'}\sum_{m+m'=n-P_0}d_m(t')d_{m'}(t')\right]$$

$\xi, \gamma_n=(n+P_0-\xi)/(n_0+P_0+\xi)$ and $\beta_n=n+P_0-\xi$ are defined in (22). The quadratic part exists when $n\geqslant P_0$.

Proof of Proposition A. (i) For fixed n, $m+m'=n-P_0$, $n\geqslant P_0$, the sum

$$\gamma_m+\gamma_{m'}=\frac{m+P_0-\xi}{m+P_0+\xi}+\frac{m'+P_0-\xi}{m'+P_0+\xi}$$

is minimum when $m=0$ (respectively, $m'=0$) and $m'=n-P_0$ (respectively, $m=n-P_0$)

$$\inf(\gamma_m+\gamma_{m'})=\frac{P_0-\xi}{P_0+\xi}+\frac{n-\xi}{n+\xi}$$

(ii) Then we have

$$\gamma_m + \gamma_{m'} - \gamma_n > \inf(\gamma_m + \gamma_{m'}) - \gamma_n = \frac{P_0 - \xi}{P_0 + \xi} + \frac{n - \xi}{n + \xi} + \frac{n + P_0 - \xi}{n + P_0 + \xi}$$

The numerator of the last expression can be written as

$$(P_0 + \xi)(2P_0^2 - 3P_0\xi - \xi^2) + (n - P_0)(P_0 - \xi)(3P_0 + 2\xi)$$
$$+ (n - P_0)^2(P_0 - \xi)$$

As $n \geqslant P_0 > \xi$, a sufficient condition for $\gamma_m + \gamma_{m'} - \gamma_n$ to be positive is

$$2P_0^2 - 3P_0\xi - \xi^2 > 0$$

or

$$\rho > \rho_2 = (1/4)(3 + \sqrt{17})$$

where $\rho = P_0/\xi$.

(iii) Let $\rho > \rho_0$. We prove by induction Proposition A. As $d_m(t')$ and $d_{m'}(t')$ decrease at least like $e^{-\gamma_m t'}$ and $e^{-\gamma_{m'} t}$ (recurrence hypothesis), then

$$e^{\gamma_n t'} d_m(t') d_{m'}(t')$$

is exponentially decreasing like

$$\exp\left[-t(\gamma_m + \gamma_{m'} - \gamma_n)\right]$$

whence the result. ■

Proof of Proposition B. It is a succession of small lemmas. Proofs are straightforward. We recall that

$$n \in \{n\}_k \Leftrightarrow (k-1)P_0 \leqslant n \leqslant kP_0 - 1$$

Lemma (i). We have

$$\gamma_n \geqslant \gamma_0 = \frac{\rho - 1}{\rho + 1}$$

Lemma (ii). We have

$$\frac{k(\rho - 1)}{\rho + 1} > \frac{k\rho - 1}{k\rho + 1} \qquad \text{if} \quad \rho > \rho_k$$

N.b.: If $n \in \{n\}_2$, d_n decreases like $\inf[2(\rho - 1)/(\rho + 1), (2\rho - 1)/(2\rho + 1)]$ and Proposition B is true if $k = 2$.

(iii) Let

$$C_{k_1,k_2}(\rho) = \left[\frac{k_1\rho - 1}{k_1\rho + 1} - \frac{k_1(\rho - 1)}{\rho + 1}\right] - \left[\frac{k_2\rho - 1}{k_2\rho + 1} - \frac{k_2(\rho - 1)}{\rho + 1}\right]$$

it is an increasing function of ρ and $C_{k_1k_2} + C_{k_2k_3} = C_{k_1k_3}$, $C_{k_1k_2} + C_{k_2k_1} = 0$.

We have the inequalities

$$C_{1,k_2}(\rho) > 0 \quad \text{if} \quad \rho > \rho_{k_2}$$
$$< 0 \quad \text{otherwise}$$

and more generally, $C_{k_1,k_2}(\rho) > 0 \quad \forall \rho > \rho_{k_2}$ if $k_1 < k_2$.

(iv) Let

$$\tilde{C}_{k_1,k_2}(\rho) = \frac{k_1\rho - 1}{k_1\rho + 1} - \frac{k_2\rho - 1}{k_2\rho + 1} + \frac{(k_2 - k_1)\rho - 1}{(k_2 - k_1)\rho + 1}$$

Similarly, $\tilde{C}_{k_1,k_2}(\rho)$ is an increasing function of ρ and

$$\tilde{C}_{k_1,k_2}(\rho) > 0, \qquad \forall \rho > \rho_{k_2} \qquad \text{if} \quad k_1 < k_2$$

(v) Then, we prove Proposition B recursively. We assume it is true up to $k - 1$ and show it for k.

If $n \in \{n\}_k$, d_n decreases at least like $\inf\{\exp(-\gamma_n t), d_m d_{m'}\}$ where $m \in \{n\}_{k_1}, m' \in \{n\}_{k-k_1}, k_1 < k$.

- $\exp(-\gamma_n t)$ decreases at least like $\exp[-(k\rho - 1)/(k\rho + 1)t]$ [we take the minimum value for the block $\{n\}_k$, i.e., $n = (k - 1)P_0$].
- If $\rho < \rho_{k-1}, d_m d_{m'}$ decreases at least like $\exp\{-k[(\rho - 1)/(\rho + 1)]t\}$ (recurrence hypothesis).
- If $\rho_{k-1} < \rho < \rho_{k-2}$, $\rho_{k-2} < \rho < \rho_{k-3} \ldots$, $\rho_3 < \rho < \rho_2$ and $\rho > \rho_2$ $d_m d_{m'}$ decreases either like $\exp[-f_{k_1,k}(\rho)t]$ or $\exp[-\tilde{f}_{k_1,k}(\rho)t]$ where we have the following:

Lemma (iii).

$$f_{k_1,k} = \frac{k_1\rho - 1}{k_1\rho + 1} + \frac{(k - k_1)(\rho - 1)}{\rho + 1} = C_{k_1,k}(\rho) + \frac{k\rho - 1}{k\rho + 1} > \frac{k\rho - 1}{k\rho + 1}$$

Lemma (iv).

$$\tilde{f}_{k_1,k} = \frac{k_1\rho - 1}{k_1\rho + 1} + \frac{(k - k_1)\rho - 1}{(k - k_1)\rho + 1} = \tilde{C}_{k_1,k}(\rho) + \frac{k\rho - 1}{k\rho + 1} > \frac{k\rho - 1}{k\rho + 1}$$

then for $\rho > \rho_{k-1}, \exp(+\gamma_n t)d_m d_{m'}$ decreases exponentially and

$$\exp(-\gamma_n t)\int_0^t \exp(\gamma_n t')d_m(t')d_{m'}(t')\,dt'$$

decreases like $\exp\{-[(k\rho - 1)/(k\rho + 1)]t\}$ whence the result by using Lemma (ii). ■

APPENDIX D

Bound for $\|R\|$ when $d > 3$

We set

$$\bar{d}_b(t) = d_b(t)\frac{\Gamma[b+(d+1)/4]}{\Gamma(b+1)}$$

Expression (24) for $\|R\|^2$ reads now

$$\|R\|^2 = \sum_a \sum_{b+b'=a} 2^{-a-d/2}\bar{d}_b(t)\bar{d}_{b'}(t)\frac{\Gamma(a+d/2)}{\Gamma[b+(d+1)/4]\Gamma[b'+(d+1)/4]}$$

$$< \Lambda^2 \sum_{b,b'} \bar{d}_b(t)\bar{d}_{b'}(t)$$

$$= \left[\Lambda \sum_b \lambda_b d_b(t)\right]^2$$

where Λ is a constant and

$$\lambda_b = \frac{\Gamma[b+(d+1)/4]}{\Gamma(b+1)}$$

or, written differently,

$$\|R\| < \Lambda \sum_n \lambda_n d_n(t)$$

where we have done the substitution $b = (n + P_0)/\xi$ and $d_b(t)$ is rewritten as $d_n(t)$. We prove that

$$\tilde{N}(t) = \sum_n \lambda_n d_n(t)$$

goes exponentially to zero for long times.

From (22) we get

$$|d_n|\lambda_n < \exp(-\gamma_{n_0}t)\left[\lambda_n|d_n(0)| + \frac{1}{\beta}\int_0^t \exp(\gamma_{n_0}t') \sum_{m+m'=n-P_0} \times |d_m|\,|d_{m'}|\lambda_m\lambda_{m'}\left(\frac{\lambda_n}{\lambda_m\lambda_{m'}}\right)dt'\right]$$

As

$$\frac{\lambda_n}{\lambda_m\lambda_{m'}} < \frac{\lambda_n}{\lambda_0\lambda_{n-P_0}} < \frac{\lambda_{P_0}}{\lambda_0^2}$$

we have

$$\tilde{N}(t) < \exp(-\gamma_{n_0} t)\left[\tilde{N}(0) + \frac{1}{\bar{\beta}}\frac{\lambda_{P_0}}{\lambda_0^2}\int_0^t \exp(\gamma_{n_0} t')\tilde{N}^2(t')\,dt'\right]$$

and the end of the proof is similar to that for $d \leqslant 3$ with $N(t)$ and $\bar{\beta}$ is replaced by $\bar{\beta}\lambda_0^2/\lambda_{P_0}$. Results for $|d_n(t)|/|d_n(0)|$ and $N_{nl}(t)/N_l(t)$ [Eqs. (27a, b)] are unchanged.

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